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EXPERIMENTAL PHYSICS SCRIPTS

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Contents

1 Kinematics

THIS CHAPTER covers basics of kinematics. It introduces the concepts of motion without relating to physical mechanisms causing it. Another simplification is that only particles are considered, i.e. motion of point-like objects are considered. This point may be associated with the center-of-mass of a bigger object. In this way, any position of an object may easily be characterized by its coordinates.

Three fundamental concepts of kinematics are *displacement*, *velocity*, and *acceleration*.

1.1 Motion along a line

IN THIS LECTURE we make one more simplification and consider only one-dimensional motion along a straight line. Thus, we will not need the concept of vectors, which will be introduced in one of the succeeding lectures. Nonetheless, we still need to distinguish two possible directions one may move along a line, let say to the right and to the left.

By selecting an appropriate coordinate system, we may fix the origin, initial and current positions, and positive and negative directions.

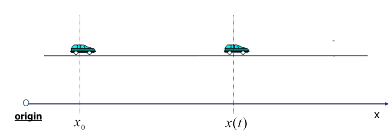

Figure 1.1: Coordinate system

1.1.1 Constant velocity

Let us first consider the simplest constant-velocity case. Let us select arbitrarily a position $x(t)$ at a time instant t. Given a time interval Δt to move, the final position will be $x(t+\Delta t)$. The displacement Δx during Δt is $x(t+\Delta t) - x(t)$. By definition, the average velocity $\bar{v}$ in the time interval Δt is

$\bar{v}$: By the over-bar symbol we typically denote the time average, if not stated otherwise.

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x(t+\Delta t) - x(t)}{\Delta t} \tag{1.1}$$

For the constant-velocity case, $\bar{v}$ is irrespective of the choice of Δt or time t and the bar symbol may be omitted.

- **EXPERIMENT:**
  Demonstration of the independency of v on Δx

  EVALUATION:

1.1.2 *Varying velocity*

If velocity is not constant, the average velocity becomes a function of both t and Δt. In this case, a useful concept is an *instantaneous* velocity, which is obtained by taking the limit of $\Delta t \to 0$:

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \equiv \dot{x} \qquad (1.2)$$

It turns out that in the limit of small Δt, $\dot{x}$ approaches a constant value, which is not a function of Δt anymore, but t only. This value we call *instantaneous* velocity. If velocity is changing along a trajectory, then, quite generally, $v \neq \overline{v}$.

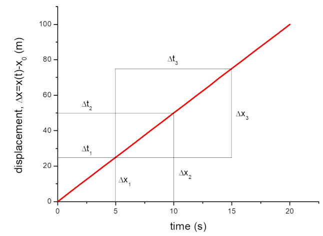

Figure 1.2: Constant-velocity motion

- **EXPERIMENT:**
  Demonstration of the dependency of v on Δx

  EVALUATION:

- **EXAMPLE 1.1.1:**
  Calculate v and $\overline{v}$ for the displacement graph shown in Fig. 1.3.

  SOLUTION:
  Let us denote the time when the velocity changes from v_1 to v_2 by t_1. On the time interval $t \leq t_1$ $v = \overline{v}$. For $t > t_1$ we need to perform simple arithmetic average with the weights t_1/t and $(t - t_1)/t$:

$$\overline{v}(t > t_1) = \frac{t_1}{t} v_1 + \frac{t - t_1}{t} v_2 \qquad (1.3)$$

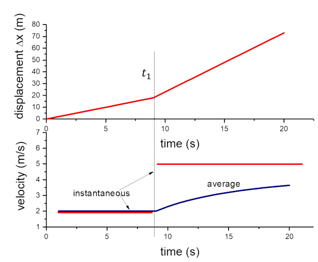

Figure 1.3: Speeding up

- **EXAMPLE 1.1.2:**
  Find $\overline{v}$ and $\overline{s}$ for an object moving for 1 h to the right with 60 km/h and then for one 1 h to the left with 40 km/h.

  SOLUTION:
  Here simple arithmetic mean applies: $\overline{v} = \frac{1}{2}v_1 + \frac{1}{2}v_2 = 10$ km/h, $\overline{s} = \frac{1}{2}s_1 + \frac{1}{2}s_2 = 50$ km/h.

- EXAMPLE 1.1.3:
  What would be the difference, if now not identical time intervals, but equidistant separations are considered. The average velocity $\overline{v}$ and the average speed $\overline{s}$ may not coincide. Find $\overline{v}$ and $\overline{s}$ for an object moving from A to B with 60 km/h and back with 40 km/h.

  SOLUTION:
  First, we find times t_1 and t_2 needed for paths AB and BA:
  $t_1 = L/|v_1| = t_1/s_1$ and $t_2 = L/|v_2| = t_2/s_2$. Thus,
  $\overline{v} = \frac{t_1}{t_1+t_2}v_1 + \frac{t_2}{t_1+t_2}v_2$, $\overline{v} = \frac{1}{t_1+t_2}\left(\frac{L}{|v_1|}v_1 + \frac{L}{|v_2|}v_2\right) = 0$. The same result could be obtained easier by noting that $\overline{v} = \Delta x/\Delta t$, and the total Δx is zero. If we apply the same equation to find $\overline{s}$, the result is $\frac{1}{\overline{s}} = \frac{1}{2}\left(\frac{1}{s_1} + \frac{1}{s_2}\right)$. This is called *harmonic* mean. The result is then $\overline{s} = 48$ km/h.

1.1.3 *Constant acceleration*

Let us now consider not displacement vs. time, but velocity vs. time coordinates. In analogue with Eq. 1.3 we may introduce a average acceleration

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v(t+\Delta t) - v(t)}{\Delta t} \tag{1.4}$$

If velocity varies linearly with time, $\bar{a}$ is irrespective of Δt and t and is simply equal to a single-valued $a = \dot{v} = \ddot{x} \equiv \frac{d^2x}{dt^2}$.

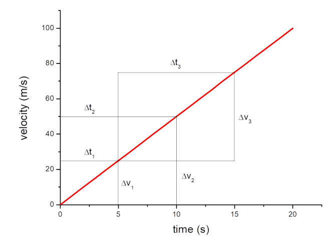

Figure 1.4: Constant-acceleration motion

1.1.4 *Complex motion*

Rally Paris-Dakar is one of the prominent examples of complex motion with varying velocity and acceleration. Fig. 1.5 shows an example where the first derivative of the car position yields instantaneous velocities and the second derivative (first derivative of velocity) instantaneous accelerations.

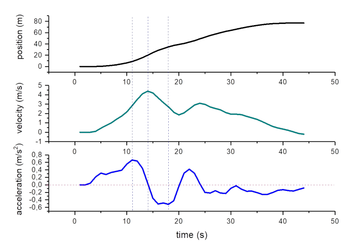

Figure 1.5: An example of complex motion

- EXPERIMENT:
  Demonstration showing first and second derivatives of $x(t)$ and comparing the derived $\ddot{x}$ with the directly measured one.

  EVALUATION:

1.1.5 *Constant acceleration kinematic equations*

The following set of equations can be used to analyse different situations with one parameter being unknown:

-	Equation	Missing
1	$v = v_0 + at$	Δx
2	$x = x_0 + v_0 t + \frac{1}{2}at^2$	v
3	$v^2 = v_0^2 + 2a(x - x_0)$	t
4	$x = x_0 + \frac{1}{2}(v + v_0)t$	a
5	$x = x_0 + vt - \frac{1}{2}at^2$	v_0

Table 1.1: Basic kinematic equations

- EXERCISE 1.1. 1:
  Derive all equations in Table 1.1

- EXAMPLE 1.1.4:
  Derivation of the equation in line 2. One may think that if one replaces v in $x = x_0 + vt$ by $v = v0 + at$ on gets the answer, but this results in $x = x_0 + v_0 t + at^2$. What causes the problem?

  SOLUTION:
  The problem is that v is the instantaneous velocity, but in $x = x_0 + vt$ it has the meaning of the average velocity, i.e. we should write $x = x_0 + \overline{v}t$. For the constant-acceleration case we considering, $\overline{v} = v_0 + \frac{1}{2}(v(t) - v_0) = \frac{1}{2}(v + v_0)$. Thus, $x = x_0 + v_0/2 + v(t)/2$. With $v(t) = v_0 + at$ one gets the equation in line 2 of Table 1.1.

- EXAMPLE 1.1.5:
  Derivation of the equation in line 3. A U-Bahn needs most quickly approach station B which is 2 km apart from station A. The maximal acceleration tolerated by the passengers is 2 m/s^2. What would be the maximal speed attained by the U-Bahn?

  SOLUTION:
  Obviously, the train need to move with the maximal acceleration half-time and maximal deceleration second half. The velocity along the acceleration path will be $v = v_0 + at$. The path is $x = x_0 + v_0 \left(\frac{v-v_0}{a}\right) + \frac{1}{2}a\left(\frac{v-v_0}{a}\right)^2$. By simplifying this equation one gets the one on line 3. The result at half distance is $v = 63.2$ m/s, which is about 227.6 km/h.

- EXPERIMENT:
  A ball falls down from a height h with zero initial velocity. After each collisions the speed changes by a factor of $a < 1$. Find the total time T that ball will stop bouncing. The coefficient a can be determined by measuring the speeds before and after collision. It may also be proven that a is relatively constant for different speeds.

EVALUATION:
Let us consider the problem in small steps just for the instructive purpose.

- Time t_0 to reach the ground is found from $0 = h - gt_0^2/2$. Thus $t_0 = \sqrt{2h/g}$. Compile a plot $h - t^2$ and determine g
- Just before meeting the ground velocity will be $v_1 = -gt_0 = -\sqrt{2hg}$.
- Bouncing upwards will occur with velocity of av_1.
- One needs now to establish the coefficient a. To find a one can either experimentally measure to which height h_1 will the ball jump if it is released from a height h OR this can be done by directly measuring the velocities before and after collision.
- Time t_{1u} to reach the upper part of the trajectory is found from $0 = av_1 - gt_{1u}$, i.e. $t_{1u} = av_1/g$.
- The height h_1 the ball approaches is $h_1 = 0 + av_1t_{1u} - \frac{1}{2}gt_{1u}^2 = \frac{1}{2}\frac{a^2v_1^2}{g}$.
- The coefficient is found to be $a = \sqrt{h_1/h}$.
- The question is what is time t_{1d} to reach the ground back. Would it be equal to t_{1u} or different? The fall time t_{1d} is found from $0 = h_1 - gt_{1d}^2/2$. One finally gets $t_{1d} = \sqrt{\frac{2h_1}{g}} = \frac{av_1}{g}$. This shows the symmetry between the rising and falling times. Experiment proving the equality of fall and rise times.
- Thus, the total time t_1 during this bouncing period is $t_1 = \frac{2av_1}{g}$.
- It is easy to see that $t_2 = \frac{2a^2v_1}{g}$ and $t_k = \frac{2a^kv_1}{g} = 2a^k\sqrt{\frac{2h}{g}}$.
- The total time T is found as $T = t_0 + \sum_{k=1}^{k=\infty} t_k$. The sum of geometric progression $\sum_{k=1}^{k=\infty} a^k = \sum_{k=0}^{k=\infty} a^k - 1 = \frac{a}{1-a}$. Thus, $T = \sqrt{2h/g} + \frac{2a}{1-a}\sqrt{2h/g} = \sqrt{2h/g}\frac{1+a}{1-a}$.

The equivalence of the rising and falling times is a consequence of the fact that the laws of motion in classical mechanics exhibit *time reversibility*, i.e. the equations are invariant under a change in the sign of time.

- EXAMPLE 1.1.6:
The time reversibility may be seen by considering the same bouncing problem without energy loss.

SOLUTION:
Indeed, the equation describing the upward trajectory (line 2 of Table 1.1) is $h = 0 + v_0 t - gt^2/2$. The downward trajectory (line 5 of Table 1.1) is described by $0 = h - v_f t + gt^2/2$. If we change $t \to -t$ in the last equation and rearrange it, the we arrive to $h = 0 - v_f t - gt^2/2$ which is exactly the same as one for the upward trajectory.

1.1.6 *Kinematic equations from calculus*

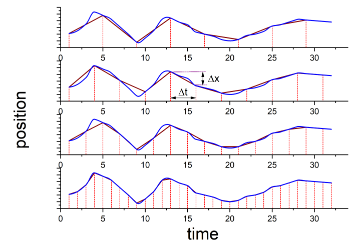

Figure 1.6: Taking limit $\Delta x \to dx$

The total displacement or a complex motion can be found by dividing the path on parts of equidistant intervals in time, Δt_i, and assuming that within this interval of time the object moved with a constant average velocity v_i. Then, the total displacement $X = \sum_i^N v_i \Delta t$. It is easy see from Fig. 1.6 that shorter is the time interval Δt (correspondingly larger N), closer it the result to the real displacement. Taking limit $\Delta t \to dt$ one finds $dx = vdt$, where v is now instantaneous velocity. Summation is now replaced by integration and one gets

$$\int_{x_0}^{X} dx = \int_0^t vdt = \int_0^t (v_0 + at)dt \tag{1.5}$$

which results in the already famous kinematic equation $X - x_0 = v_0 t + at^2/2$. Similarly, one may find velocity as

$$\int_{v_0}^{v_f} dx = \int_0^t adt \tag{1.6}$$

1.1.7 *List of experiments*

1. Constant velocity case: v is irrespective Δx
2. Varying velocity case: v depends on Δx
3. Complex motion: x, v, a
4. Changing speed upon collision, determining the ratio
5. Proving the h vs. t^2 law
6. Proving that rise and fall times are identical
7. Find time for a ball released from h to stop bouncing

1.2 *Scalars and vectors*

IN THIS LECTURE we recall the basics of scalars and vectors.

1.2.1 *Scalars and vectors*

A scalar quantity is specified by a single value with an appropriate unit and does not reflect any direction (mass, volume, temperature). A vector quantity has in contrast both magnitude and direction (velocity, flux, force).

Vectors might be useful in many instances, for example in describing trajectories as shown in Fig. 1.7. They may be subdivided to many small sections each captured by a vector. The total displacement then is the vector sum of individual ones.

Vectors can be added as geometrically shown in Fig. 1.8. In this way a negative vector turns out to be one giving zero upon adding with the original vector. Summation of vectors complies with the following laws:

Commutative law $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

Associative law $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

Substraction law $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

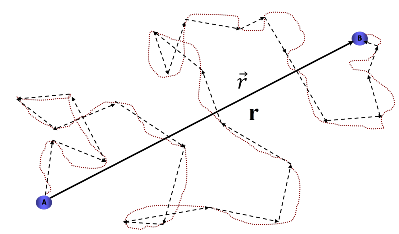

Figure 1.7: Trajectory

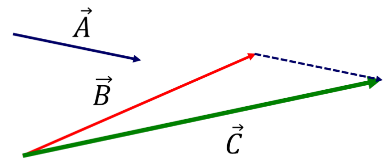

Figure 1.8: Adding two vectors

Cartesian coordinate system ia right-handed coordinate system. The index finger , the middle finger , and the thumb now give the alignments of the x-, y-, and z-axes, respectively.

1.2.2 *Vector components*

1.2.3 *Scalar and vector products*

- EXPERIMENT:
  A piece of mass is first pulled along a table horizontally. Afterwards, it is pulled under certain angle. The work done is found as a scalar product of the force and displacement.

  EVALUATION:
  The constancy of the force may be secured by hanging another mass on one end of the chord.

- EXPERIMENT:
  A wire carrying electric current is placed in a magnetic field. The wire is forced perpendicularly to both magnetic field and current directions.

  EVALUATION:
  Instead of wire one may use an electron beam.

1.2.4 *List of experiments*

1. Demonstration of scalar product (work)
2. Demonstration of vector product (Lorenz force)

1.3 Motion in 2D and 3D

IN THIS LECTURE we extend motion to higher dimensions to consider curvilinear pathes. We us for that the concept of vector introduced in the preceding section.

1.3.1 Average and instantaneous velocities

Fig. 1.9 shows the initial and final positions of a material point, the path connecting these two points, and also two radius vectors in an arbitrarily selected coordinate system describing these points. The average velocity $\vec{v}_{avg} \equiv \bar{\mathbf{v}}$ (note that it is as well a vector quantity) is defined as

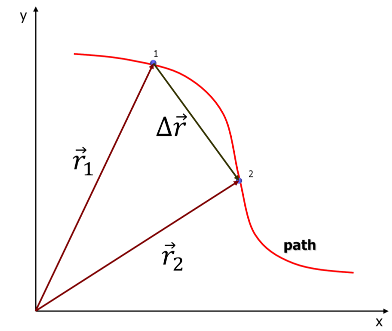

Figure 1.9: Displacement

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} \tag{1.7}$$

As any vector, it may represented by its vector components.

The average velocity is not a function of path connecting initial and final positions.

The instantaneous velocity $\vec{v}$ is defined as the limit of the average velocity at $\Delta t \to 0$, i.e.

$$\vec{v} = \frac{d\vec{r}}{dt} \tag{1.8}$$

Instantaneous velocity is always tangential to the path. This statement can most simply be proven by considering motion along a circular path. The radius vector of an object $\vec{r} = R\hat{r}$. Hence,

$$\vec{v} = \frac{d\vec{r}}{dt} = R\frac{d\hat{r}}{dt} + \hat{r}\frac{dR}{dt} \tag{1.9}$$

The last term on right hand side of Eq. 1.9 is zero because R is constant. Let us now check the direction of $d\hat{r}/dt$. For this, we find the dot product $\hat{r} \cdot \frac{d\hat{r}}{dt}$:

$$\frac{d(\hat{r}\cdot\hat{r})}{dt} = 2\hat{r}\cdot\frac{d\hat{r}}{dt} = 0 \tag{1.10}$$

Indeed, because $\hat{r}\cdot\hat{r} = const$ the derivative is zero. Thus, Eq. 1.10 proves that $d\hat{r}/dt$ is perpendicular to $\hat{r}$.

- EXPERIMENT:
  Sharpening of a knife using rotating disk. All sparks are moving tangential to the disk.

  EVALUATION:

1.3.2 *Average and instantaneous accelerations*

By definition the average acceleration is

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} \tag{1.11}$$

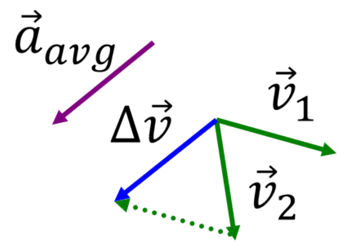

Figure 1.10: Average acceleration

and its direction coincides with that of the average velocity as shown in Fig. 1.10.

Let as consider the instantaneous acceleration, which is defined as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{1.12}$$

Quite generally, $\vec{a}$ has both normal and tangential components to the path. Indeed,

$$\frac{d\vec{v}}{dt} = \frac{d(v\hat{s})}{dt} = v\frac{d\hat{s}}{dt} + \frac{dv}{dt}\hat{s} \tag{1.13}$$

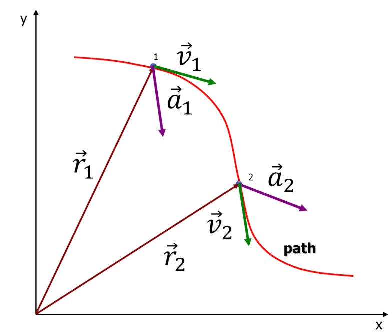

Figure 1.11: Instantaneous acceleration

In the spirit of Eq. 1.10 it is easy to see that $d\hat{s}/dt$ is normal to s and may be denoted as $\hat{n}(\perp \hat{s})$. Hence, Eq. 1.13 may be rewritten as

$$\vec{a} = a_t\hat{s} + a_n\hat{n}, \tag{1.14}$$

where a_t and a_n are the tangential and normal components of the acceleration vector, respectively.

- EXAMPLE 1.3.1:
  By considering circular motion establish the meaning of a_n.

  SOLUTION:
  Let us for simplicity consider v being constant. The first term in Eq. 1.13, $vd\hat{s}/dt$ may be found in the following way. $\vec{v} = d\vec{r}/dt = Rd\hat{r}/dt$, $\omega = d\varphi/dt = v/R$. Because $\vec{v} = v\hat{s}$, the comparison yields $d\hat{r}/dt = \omega\hat{s}$. Let us consider $d(\hat{s}\cdot\hat{r})/dt$. It is zero on the one hand. On the other hand it yields $\hat{r}d\hat{s}/dt = -\hat{s}d\hat{r}/dt$. Hence, $d\hat{s}/dt = -\omega\hat{r}$. Finally, $a_n = -v\omega$.

1.3.3 *Equations of motion in vector form*

We may consider as example two most important equations,

$$\vec{v} = \vec{v}_0 + \vec{a}t \tag{1.15}$$

$$\vec{r} = \vec{r}_0 + \vec{v}t + \frac{1}{2}\vec{a}t^2 \tag{1.16}$$

These two equations can be rewritten in the component form.

- EXPERIMENT:
  To illustrate that the equations of motion for each component are disentangled one may show the "schnips" experiment, where one ball is given a horizontal impulse, while the second one starts simultaneously free fall

  EVALUATION:
  Both balls reach the ground at the same time instance.

The best example for this independency is a projectile. For horizontal motion one finds

$$x = x_0 + v_{0x}t = x_0 + v_0 \cos(\theta_0)t \quad (1.17)$$

For the vertical projection there are two equations

$$y = y_0 + v_0 \sin(\theta_0)t - gt^2/2 v_y = v_0 \sin(\theta_0)t - gt \quad (1.18)$$

We are interested to find the equation of path for the projectile, i.e. a function $y = y(x)$. Expressing t from Eq. 1.19, the required equation results as

$$y - y_0 = \tan(\theta_0)(x - x_0) - \frac{g}{2}\left(\frac{g(x - x_0)}{v_0 \cos(\theta_0)}\right)^2 \quad (1.19)$$

With the equation of path one may find some quantities of interests as shown by the following examples.

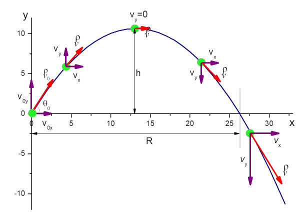

Figure 1.12: Projectile

- EXAMPLE 1.3.2:
  Find the height h at the highest point of the projectile.

  SOLUTION:
  To find h one needs to differentiate y with respect to x and equation to zero. In this way one finds $x = v_0^2 \sin(\theta_0)\cos(\theta_0)/g$ as the x-coordinate at the extremum. Substituting this to find y results in $h = (v_0 \sin(\theta_0))^2/2g$

- EXAMPLE 1.3.3:
  Find the angle θ_0 such that the longest distance is provided.

  SOLUTION:
  Because of the symmetry of the projectile and using the result of the preceding example the distance is $x = 2v_0^2 \sin(\theta_0)\cos(\theta_0)/g$. By differentiating this equation with respect to θ_0 one finds $\cos^2(\theta_0) - \sin^2(\theta_0) = 0$. Hence at $\theta_0 = 45°$ the distance will be longest. The later is also graphically seen from Fig. 1.13.

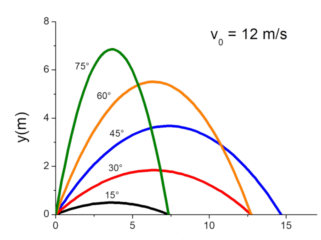

Figure 1.13: Projectile

- EXPERIMENT:
  Shooting a monkey.

  EVALUATION:
  Monkey is hit

- EXPERIMENT:
  Water stream.

  EVALUATION:
  Proving the longest distance at $\theta_0 = 45°$.

1.3.4 *Circular motion in component form*

The radius vector of a particle moving along a circle with a constant angular velocity $\omega = d\phi/dt$ in the polar coordinates system is

$$\vec{r} = R\cos(\omega t)\hat{i} + R\sin(\omega t)\hat{j} \tag{1.20}$$

Taking time derivative the particle velocity results as

$$\vec{v} = -R\omega\sin(\omega t)\hat{i} + R\omega\cos(\omega t)\hat{j} = \omega\left(-R\sin(\omega t)\hat{i} + R\cos(\omega t)\hat{j}\right) \tag{1.21}$$

If we introduce the unit tangential vector $\hat{t}$, Eq.1.21 simplifies to

$$\vec{v} = R\omega\hat{t} \tag{1.22}$$

By taking the time derivative of Eq.1.21, acceleration is obtained as:

$$\vec{a} = -\omega^2\left(R\cos(\omega t)\hat{i} + R\sin(\omega t)\hat{j}\right) \tag{1.23}$$

It is readily seen that the angular acceleration is antiparallel with the radius vector. By finding its magnitude and with the direction established, the angular velocity for circular motion is

$$\vec{a_r} = -\omega^2 R\hat{r} = -\frac{v^2}{R}\hat{r} \tag{1.24}$$

If the angular velocity is not constant., i.e. $\omega = \omega(t)$, one finds

$$\vec{a} = \vec{a_r} + \frac{dv}{dt}\left(-\sin(\omega t)\hat{i} + \cos(\omega t)\hat{j}\right) \tag{1.25}$$

The second term on the RHS of Eq.1.25 is the tangential acceleration

$$\vec{a_t} = \frac{dv}{dt}\hat{t} \tag{1.26}$$

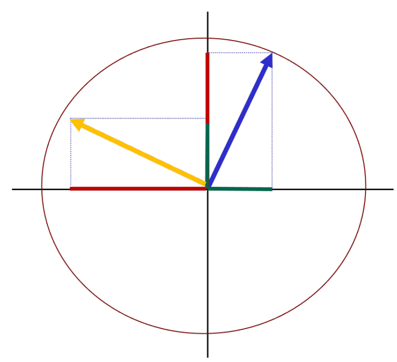

Figure 1.14: Figure demonstrating that the vectors given by Eqs. 1.20 and 1.21 are orthogonal.

2 The laws of motion

In the previous we have discussed laws of motion under given initial conditions, for fixed trajectories and/or given velocities and accelerations. In this chapter we attempt to answer what caused motion, why under certain condition objects move along particular trajectories, or what we can learn about those objects if we may follow their trajectories.

2.1 The Newton's laws

In this lecture we recall three most important Newton's laws forming the basis for modern classical mechanics. The introduction of these laws was a conceptual advancement over the earlier theories, such as by Aristotle. In the older theories, even thought the forces were causing motions of objects, they were treated as the necessary condition for any type of motion. Accordingly, if a force stopped acting on an object, the object thought to come to rest. Any instances when objects were moving without action of any forces, like arrow leaving a bow, were discussed in an awkward and inconsistent manner. The Newtonian mechanics treated the forces conceptually differently, i.e. as causing *the changes* in motion.

Before introducing the Newtonian mechanics, it can be mentioned that one may distinguish two type of macroscopic forces, contact forces and field forces. Two push a car along a road one needs to apply a contact force. On the other hand, a ball is "attracted" by the Earth by applying a contact-less, distant force, the gravity force. To date, all types of forces one may experience in the every-day life are the results of just four fundamental forces: gravitational, electromagnetic, strong and weak nuclear forces.

2.1.1 Newton's first law

The Newton's first law states that object will continue to be in rest or will continue its motion with constant velocity unless it is acted by an unbalanced force. Therefore, sometime this law is also referred to as the law of inertia. A critical advancement over ancient theories is considering the rest or motionless state to be qualitatively identical with the states of movement with constant velocity. This makes all inertial coordinate systems, i.e. that moving with respect to each other with constant velocities, to be equivalent. If in one of them, all forces acting on some object balance to zero, for the external observers in other inertial reference systems the net force acting on the object as well will appear to be zero.

Zero net force means that all force components are equal to zero as well.

- EXPERIMENT:
  A ball rolling down an inclined surface continues to roll with a constant velocity after leaving the inclined surface. It will, however, come to rest if enters a rubber sheet.

  EVALUATION:
  Along horizontal table the gravity force is zero. For a table covered with some soft (rubber) film the frictional force changes the otherwise constant velocity.

- EXPERIMENT:
  The two masses are suspended with threads as shown in Figure 2.1. If to pull the lowest thread down as shown by the arrows, which of the three threads will break down?

  EVALUATION:
  Depends on how quick to pull down. If slowly, then tension in the uppermost thread will be highest (pulling force plus gravitational force of the two masses) and it breaks down first. If quickly enough, then the two masses having inertia will tend to keep their rest state and, for this short interval of time, do not transmit tension to the upper threads. Hence, tension quickly develops only in the lowest threads and it breaks down.

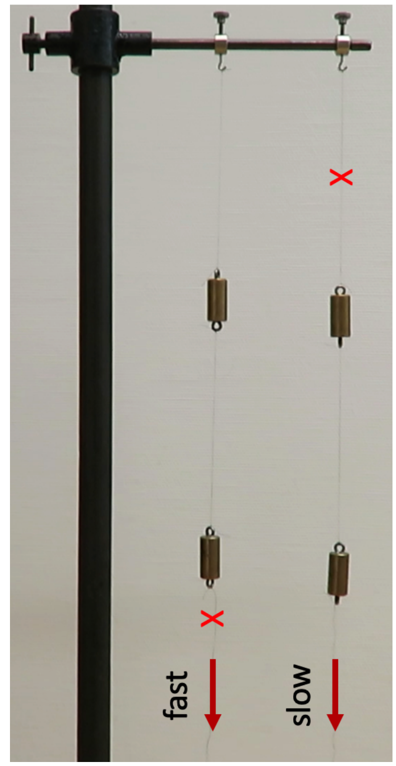

Figure 2.1: Two masses suspended with threads

The Newton's firs law allows designing very simple devices for measuring forces. For example, if you fix a spring at one end and pull it from another end so that you extend the spring by some amount Δx, then you will feel that you need to apply some constant force F to keep Δx constant. Because the spring or you hand do not move, the Newton's law says that the force you apply needs to be exactly compensated by another force. In this particular case, this is the so-called spring force acting in opposite direction. Already in 17th century id was established by the French philosopher Robert Hooke that, at least for small extensions Δx, the spring force F_s is a liner function of Δx:

$$F_s = -k\Delta x \tag{2.1}$$

where k is a spring constant. Thus, any spring can easily be calibrated and used to measure forces.

2.1.2 *Newton's second law*

The Newton's second law says that the rate of change of velocity of an object, i.e. acceleration, is proportional to the net force acting on the object and is inversely proportional to its mass. Because force is a vector quantity, acceleration thus occurs in the direction of the applied force. In fact, one may consider the second Newton's law as either definition of force or mass. Indeed, the force needed to accelerate 1 kg mass to 1 m/s^2 can be defined as 1 Newton. Alternatively, mass can be considered as measure of the resistance to acceleration - the heavier is the mass, the larger is the force needed to apply to accelerate an object to the same value.

2.1.3 *Newton's third law*

The third Newton's law states that all forces always appear in pairs. If one object exerts a force $\vec{F}$ on another object, the latter will react with the opposite force $-\vec{F}$ on the former one. A classical example of this action-reaction pair is the gravity force. If the Earth attracts a ball with a force $F_g = -mg$ conventionally assumed to be downwards, the same way the ball exerts equal in magnitude, but opposite force $F_g = mg$ to the Earth. Note, that the action and reaction forces are always applied to different objects.

- EXERCISE 2.1. 1:
  If both, the Earth and ball, attract each other with identical forces, why do we say that the ball is falling down to the Earth and not the Earth is falling dawn to the ball?

- EXERCISE 2.1. 2:
  When tram stops, the passengers are typically fall backwards, not forwards in accord with the law of inertia. Why?

2.1.4 *Selected problems*

A general scheme for solving mechanical problems on the basis of three Newton's equation comprises several steps:

1. Isolating the object(s) of interest
2. Drawing of all forces acting on object
3. Selecting a suitable coordinate system
4. Writing down the second Newton's law in component form
5. Solving the system of resulting equations
6. Checking yourselves (dimension, limiting cases, etc.)

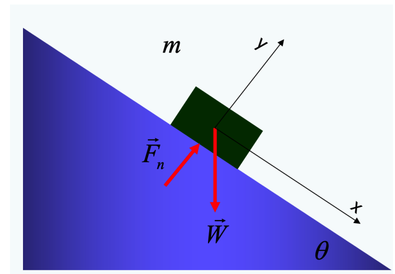

Figure 2.2: An object sliding dawn a frictionless surface

Let us consider a trivial example of an object sliding down a frictionless surface as shown in Figure2.2 and apply the scheme outlined above.

1. We are interested in behavior of the object on the surface.
2. There are only two forces acting, the gravity force down F_g and the normal force F_n. Note that the normal force is perpendicular to the surface. One need to introduce it to exclude movement of the object in the direction perpendicular to the surface by compensating the respective gravity force component. Also, the normal force cannot be selected to be in the direction opposite to the gravity force. Otherwise, the object would not move at all.
3. Because we are interested in how the object will slide down the surface, it is convenient to select the coordinate system with the axes parallel and perpendicular to the surface.
4. In the perpendicular direction there is no motion. Hence,

$$F_n - mg\cos(\theta) = 0 \tag{2.2}$$

   In the parallel direction only the component of the gravity force is responsible for acceleration:

$$mg\sin(\theta) = ma \tag{2.3}$$

5. The system of resulting equations is too trivial and is already the solution
6. For example, acceleration is found to be irrespective of mass. Does it make sense? Yes, because we do not consider and friction or drag forces. In this sense, it is reminiscent of the situation with falling objects under gravity force under vacuum atmosphere. Their acceleration is indeed irrespective of mass. As another limiting case, if θ is equal to zero, no acceleration should result, which is also correct.

Another classical example is the so-called Attwood machine representing two masses connected with a thread over a pulley as shown in Figure 2.3. Let us consider first how the masses will move. The effect of the gravity force due to the two masses is to create a tension T in the thread. Because this is physically one string and there is no friction between the pulley and the thread, tension will be the same over the entire thread. The Newton's second law applied to the two masses results, thus, in

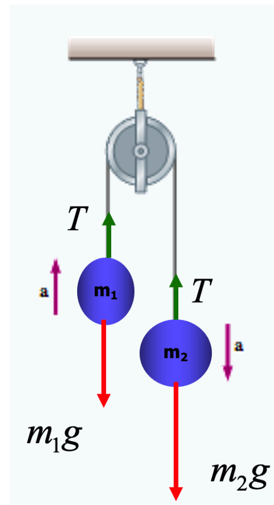

Figure 2.3: Attwood machine

$$T - m_1 g = m_1 a \tag{2.4}$$
$$T - m_2 g = -m_2 a \tag{2.5}$$

Here we used the fact that tension is not zero, hence the masses will move with identically, but in opposite directions. The system can be solved either for tension or for acceleration as

$$a = g\frac{m_2 - m_1}{m_2 + m_1} \tag{2.6}$$
$$T = g\frac{2m_1 m_2}{m_2 + m_1} \tag{2.7}$$

A simple check is making two masses identical, m. Then, zero acceleration results as expected. Rationalizing $T = mg$ and not, for example, $2mg$, is, perhaps, less intuitive. But the first Newton's law makes it clear. Indeed, if the two masses are identical, then they do not move and the net force on them must be zero. As another limiting case, let us consider one of the masses being zero. In this case, acceleration is g and tension is zero, as expected.

- EXERCISE 2.1. 3:
  When in Eq. 2.6 we set $m_2 = 0$, then $a = -g$ as expected. However, for $m_1 = 0$ it gives $a = g$, i.e., it predict the wrong direction. Does it mean that Eq. 2.6 is not correct?

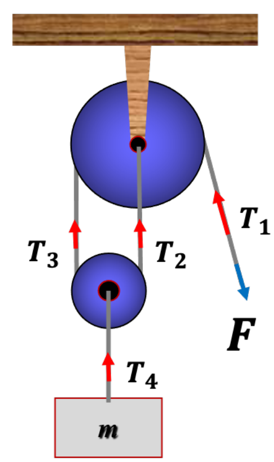

Figure 2.4: Two-pulley system

Yet another example might be very useful in every-day life and is shown in Figure 2.4. The problem to solve is what is the minimal force F one needs to apply to lift up the mass m? Once again one may use the fact that tension in the thread is identical, so that $T = T_1 = T_2 = T_3$. If thread does not move, the force F just balances tension T, $F = T_1 = T$. To find T, let us consider the lower pulley in rest. In this case, $T_2 + T_3 = 2T = mg$. Consequently, it turns that $F = mg/2$. That means that if with one pulley one needs applying the force $F = mg$, with the clever pulley system the force required is only half of the weight to lift up.

2.1.5 *Frictional forces*

If to push a heavy mass lying on a table parallel to the surface, quite counterintuitively and in contrast to the first Newton' law the mass will not move.If you increase the force you are pushing with nothing happens. If only some threshold level for the force applied is reached, then the mass starts sliding. In fact, this observation does not violate the Newton's law, but rather corroborates it. Bringing two bodies in contact (mass and table) results in the occurrence of the frictional force between them. It turns out that the frictional force opposes the external force applied and balances it, so that the body remains in rest. Moreover, the magnitude of the frictional forces increases linearly with the applied force until some maximal value is reached. Only then the net force becomes non-zero and results in the mass acceleration. The origin of the frictional force is predominantly due to microscopic surface inhomogeneities. Figure 2.5 depicts the simplest model for the emergence of friction by showing a mass m on a surface. Both mass and surface have similar spike-like surface heterogeneities. Otherwise, there is no friction between the two bodies. It is clear that, even with the frictionless surface, one needs to apply non-zero force $F = mg\sin(\theta)$ (consult the discussion of Figure 2.2) in order the mass can slide over the surface spike if initially the two spike were in contact.

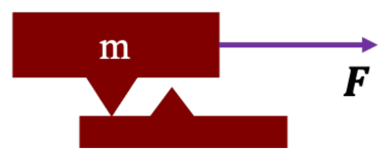

Figure 2.5: Mechanical model for friction

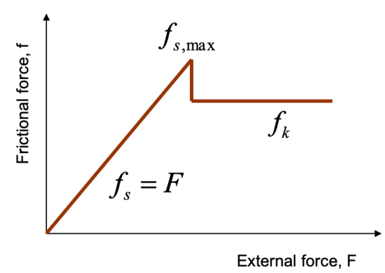

Figure 2.6: Frictional force vs. applied force

The model just discussed allows to rationalize that static frictional force f_s is proportional to the normal force F_n (as captured by mg in Figure 2.5) and to surface properties of the two bodies in contact (as modeled by $\sin(\theta)$ in Figure 2.5). Hence, quite generally the static frictional force is given by

$$f_s \leq \mu_s F_n \tag{2.8}$$

where μ_s is the coefficient of static friction. The latter is materials-dependent and can be found in tabulated form for various materials. With known F_n, μ_s is easy to measure by measuring the force needed to bring the body in motion. It is quite counter-intuitive that f_s is not a function of contact area. Indeed, one may think that frictional force is multiplicative of the contact points similar to that shown in Figure 2.5. The latter should scale with surface area. This is, in fact, true. However, the total normal force (e.g., the weight of the object) will be distributed over all contact points. Thus, if there are N contact points on the surface and at each contact point the normal force is F_n/N, then f_s results as $f_s \propto N \times (F_n/N) \propto F_n$.

If to bring the body into sliding motion, there is still friction between the bodies. However, it is found that the frictional force in this case is lower as shown in Figure 2.6. Kinetic friction is captured by the same equations as the static one, Eq. 2.8, but with the coefficient of kinetic friction $\mu_k < \mu_s$. This fact finds useful applications, such as in anti-lock braking system (ABS).

2.1.6 *Capstan equation*

It has just been argued that the magnitude of frictional force does not depend on the contact area between two bodies. This has also been demonstrated in the lectures by turning a rectangular wood cuboid on its different facets and measuring the frictional force. It turned out that the latter was irrespective of on which facet the cuboid was lying of. Quite puzzling appears, in this regard, the experiment in which a thread, supporting a piece of mass at one of the ends, was wounded around a fixed glass tube. The experiment has shown that the force, needed to balance the weight, drops exponentially with the number of wounds. Because the tube is fixed, obviously friction plays some role. But why its effect depends on the contact area or length in this case?

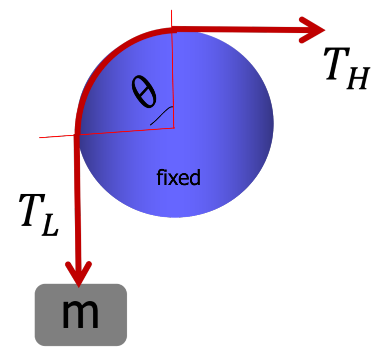

Figure 2.7: A capstan model

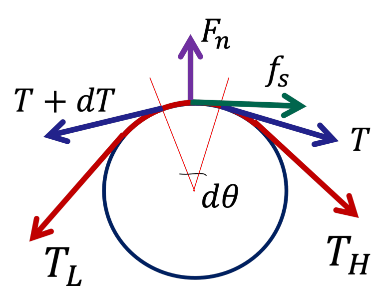

Figure 2.8: Force diagram for the model in Figure 2.7

To clarify this, let us consider the model shown in Figure 2.7. Let us fix the hold force, i.e. T_H, and find the load force, i.e. T_L, needed to balance it. If there were no friction between the thread and rod, the load T_L and hold T_H tensions in the thread would be identical and equal to mg. The rod, thus, only changes the direction of the force, but not its magnitude. With friction, however, for the mechanical equilibrium one needs to include the frictional force acting in the direction opposite to T_L. Hence, quite generally, T_H should be less than T_L. Because friction acts along the total contact line, tension becomes progressively lower towards the hold side. Let us analyse it by considering a small contact section as illustrated in Figure 2.8. It shows a small circular arc of the angle $d\theta$ formed by the thread with all forces acting on it. The mechanical balance equations, thus, are

$$(T + dT)\cos(d\theta/2) = T\cos(d\theta/2) + \mu_s F_n \tag{2.9}$$
$$(2T + dT)\sin(d\theta/2) = F_n \tag{2.10}$$

These to equations can be combined to yield

$$dT = \mu_s T d\theta \tag{2.11}$$

To find the final solution, Eq. 2.13 can be integrated

$$\int_{T_H}^{T_L} \frac{dT}{T} = \mu_s \int_0^\theta d\theta \tag{2.12}$$

resulting the Capstan or Euler-Eytelwein equation

$$T_L = T_H e^{\mu_s \theta} \tag{2.13}$$

Finally, it is an interdependency between friction and tension that leads to exponentially-fast decreasing tension in the thread with increasing contact length.

2.1.7 *Drag forces*

The drag forces are similar to the frictional forces between solid bodies. In this case, however, one of the bodies is replaced by a fluid, either gas or liquid. On of the instructive examples convincing in their occurrences is given by parachute, resulting in free fall with constant speed under the action of the gravity force. Constant speed assumes that the gravity force is balanced by one, called the drag force.

Depending on the absolute velocity (more precisely, on the Reynolds number), one may distinguish between two scenarios in which the drag force F_d is proportional to either velocity or square of velocity. The former regime is encountered for very low relative velocities and is called viscous friction regime. This phenomenon will be consider in details in one of the following sections on fluid mechanics. The latter regime can easily be quantified by using a simple mechanistic model.

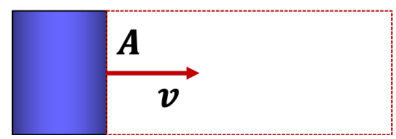

Figure 2.9: Drag force at high speeds

Let us consider a body with the mass M and cross-sectional area A moving with a high speed v in a medium containing small molecules with a density ρ. Let us also assume that the molecular velocities are notably lower than v, hence they will be compressed and collected in the front of the body as illustrated in Figure 2.9. Let us for the sake of simplicity consider free fall in the field of gravity. The second Newton's law in this case is

$$Mg = (M + m)a \tag{2.14}$$

where m is the mass collected in the front of the body and a is the acceleration of $M + m$ resulting due to the gravity force Mg. The mass collected by the body during short time interval dr can be found as

Why the fluid molecules are not subject to the free fall acceleration will be discussed in thermodynamics course.

$$dm = (A \cdot vdt)\rho \tag{2.15}$$

Let us find ma. For this one need to integrate

$$adm = (A \cdot vdt)\rho \frac{dv}{dt} = \frac{1}{2} A\rho dv^2 \tag{2.16}$$

resulting in

$$Mg = Ma + \frac{1}{2}AC_d\rho dv^2 \tag{2.17}$$

In the last equation we have introduced a phenomenological coefficient C_d accounting for the efficiency of collection of the molecules by the body. The second term on the RHS of Eq. 2.17 can be considered as the drag force F_g. Because its magnitude increases quadratically with velocity, after certain period of time it balances the gravity force and the body starts moving with a constant terminal velocity v_t. It is easily found as

$$v_t = \sqrt{\frac{2Mg}{AC_d\rho}} \tag{2.18}$$

- EXERCISE 2.1. 4:
  Estimate the typical terminal velocity for free fall with a parachute.

2.1.8 *Inertial forces*

When object performing rotational motion, e.g. a mass rotated using a thread, is considered in an inertial coordinate frame, then it is subjected to centripetal acceleration $\vec{a} = -\omega^2\vec{r}$. To give rise to this acceleration, a centripetal force

$$\vec{F}_{cp} = -m\omega^2\vec{r} \tag{2.19}$$

allowing to keep the object on the circular path must be applied. In this particular case, tension in the thread plays such role. Sometime, it becomes more convenient to work in the rotating coordinate frame. This may, e.g., simplify equations or make analysis easier. For example, in the rotating frame the object appears to be at rest and acceleration disappears. At the same time, the centripetal force is still there and in order to Newton's second law to be valid a fictitious centrifugal force $\vec{F}_{cf}$,

$$\vec{F}_{cf} = m\omega^2\vec{r} \tag{2.20}$$

balancing $\vec{F}_{cp}$, needs to be introduced. Once again, it does not exist when viewed from the inertial coordinate frame and only appears in the rotating frame.

As an example, let us consider a carousel with two masses, m and M, connected to the ropes of identical length L (see Figure 2.10). If the carousel is rotated with the angular velocity ω would the angles the masses decline from vertical be identical or different? To answer this, it is enough to consider balance of all forces acting on the body in the directional perpendicular to the rope:

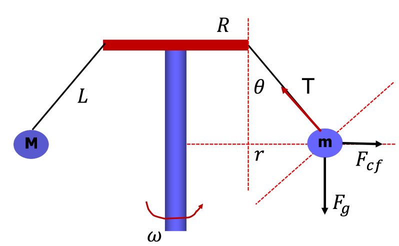

Figure 2.10: Carousel with two masses

$$F_{cf}\cos(\theta) = F_g \sin(\theta) \tag{2.21}$$

For the sake of simplicity, let us consider only small angles θ. In this case, θ is readily found

$$\theta = \frac{\omega^2 R}{g - \omega^2 L} \tag{2.22}$$

to be mass-independent.

Another example demonstrating the action of centrifugal forces is shown in Figure 2.11. When the flat cuvette will be rotated, the water interface will assume the shape shown by the black lines. To note is that there is a common point for the interfaces obtained at different rotation frequencies. Let us find now the shape of the water-air interface using the force diagram in Figure 2.12. For a small volume with mass m at the water surface to be in equilibrium Eq. 2.21 should be valid. On the other side, the variation of the interface hight, dh, with increasing distance from the rotation center, dr, is given by

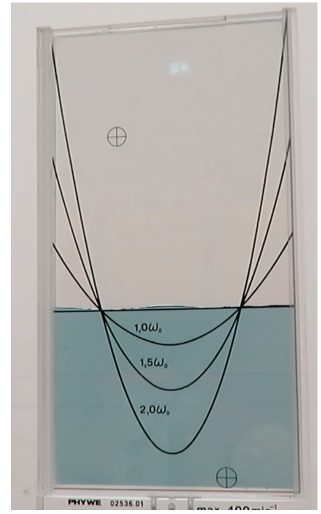

Figure 2.11: Rotating cuvette with water.

$$\frac{dh}{dr} = \tan(\theta) \tag{2.23}$$

Combining Eqs. 2.21 and 2.23 readily yields

$$h = \frac{1}{2}\frac{\omega^2}{g} r^2 \tag{2.24}$$

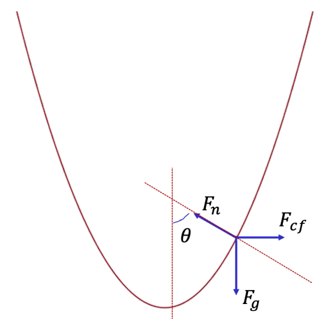

Figure 2.12: Forces on the water interface in Figure 2.11.

- EXERCISE 2.1. 5:
  Derive Eq. 2.24.

- EXERCISE 2.1. 6:
  Find the position of the crossing or common interface point obtained at different rotational frequencies in Figure 2.11.

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